Root

a- real number and n - natural number

Each solution of equations: $x^n = a$ (if any) is called n-the root of number a in the label $x = \sqrt[n]{a}$

So, symbol $\sqrt[n]{a}$ *indicates:*

1) n- the root of the real number a in all cases when it is unique

$(n \in N,\ a = 0,\ n = 2k - 1,\ k \in N,\ a \in R)$

2) Positive n- the root of the number a i n the case:

$n = 2k,\ k \in N,\ a > 0$

This definition certainly is not much clear! Go to see a few examples:

$$\sqrt[3]{27} = \sqrt[3]{3^3} = 3;\quad \sqrt[3]{\frac{1}{8}} = \sqrt[3]{\left(\frac{1}{2}\right)^3} = \frac{1}{2}$$

$$\sqrt[5]{-32} = \sqrt[5]{(-2)^5} = -2;\quad \sqrt[7]{0} = 0$$

$$\sqrt[2]{4} = \sqrt[2]{2^2} = 2;\quad \sqrt[4]{\frac{1}{16}} = \sqrt[4]{\left(\frac{1}{2}\right)^4} = \frac{1}{2}$$

$$\sqrt[4]{16} = \sqrt[4]{2^4} = 2;\quad -\sqrt[4]{16} = -\sqrt[4]{2^4} = -2$$

It is wrong to write: $\sqrt[4]{16} = \pm 2$ **REMEMBER!**

Valid:

$$\sqrt[n]{a^n} = \begin{cases} a, & n = 2k+1 \\ |a|, & n = 2k \end{cases}$$

Examples: (look out, the agreement is $\sqrt{A} = \sqrt[2]{A}$, only here we do not write 2)

$\sqrt{9} = \sqrt{3^2}$; $\sqrt[3]{2^3} = 2$

$\sqrt{(-3)^2} = |-3| = 3$; $\sqrt[3]{(-2)^3} = -2$

Remember: When you see 2, 4, 6, ... (even root) from a specific number, the solution is always positive number. When you see 3, 5, 7 ... (odd root) from a number, solution can be positive and negative number, depending on what is "in root".

$$\sqrt{(-5)^2} = |-5| = 5 \qquad \sqrt[3]{5^3} = 5$$

$$\sqrt[4]{(-7)^4} = |-7| = 7 \qquad \sqrt[3]{(-5)^3} = -5$$

$$\sqrt[6]{(-12)^6} = |-12| = 12 \qquad \sqrt[5]{\left(\frac{1}{10}\right)^5} = \frac{1}{10}$$

$$\sqrt[8]{\left(-\frac{1}{3}\right)^8} = \left|-\frac{1}{3}\right| = \frac{1}{3} \qquad \sqrt[7]{\left(-\frac{3}{5}\right)^7} = -\frac{3}{5}$$

<u>Example:</u> **For what real numbers, x is the correct value**

a) $\sqrt{x^2} = x$
b) $\sqrt[3]{x^3} = -x$
c) $\sqrt{x^2} = -x$
d) $\sqrt{x^4} = \left(\sqrt{x}\right)^4$

Solution:

a) $\sqrt{x^2} = x$ is correct only for the values of x that are greater than, or equal to zero, because

$$\sqrt{x^2} = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 0, & za \ x = 0 \end{cases} \qquad \text{Dakle } x \geq 0$$

b) $\sqrt[3]{x^3} = -x$ is only correct for x = 0. Why?

If we take that x is a negative number, for example $x = -5$, then : $x = -5 \Rightarrow$
$\sqrt[3]{(-5)^3} = -5 \neq -(-5) = +5$,
If we take that x > 0, for example x=10, then $\sqrt[3]{10^3} = 10 \neq -10$

c) $\sqrt{x^2} = -x$

$$\sqrt{x^2} = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases} \quad , \quad \text{So:} \quad x \le 0$$

d) $\sqrt{x^4} = \left(\sqrt{x}\right)^4$

$x \ge 0$ Why?

Because $\left(\sqrt{x}\right)^4$ can not be negative and must be $x > 0$,
$\sqrt{0^4} = \left(\sqrt{0}\right)^4$, so $x \ge 0$

RULES:

1) $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

2) $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

3) $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{a : b}$

4) $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

5) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[n \cdot m]{a}$

6) $\sqrt[np]{a^{mp}} = \sqrt[n]{a^m}$

EXAMPLES:

1)

a) $\sqrt{36} - 2\sqrt{25} + \sqrt[4]{16} - \sqrt[5]{32} = ?$

$$\begin{aligned} &\sqrt{36} - 2\sqrt{25} + \sqrt[4]{16} - \sqrt[5]{32} = \\ &= 6 - 2 \cdot 5 + \sqrt[4]{2^4} - \sqrt[5]{2^5} = \\ &= 6 - 10 + 2 - 2 = -4 \end{aligned}$$

b) $\sqrt{\frac{9}{4}}+\sqrt[3]{\frac{1}{8}}+\sqrt[4]{16}=?$

$=\sqrt{\left(\frac{3}{2}\right)^2}+\sqrt[3]{\left(\frac{1}{2}\right)^3}+\sqrt[4]{2^4}=$

$\frac{3}{2}+\frac{1}{2}+2=4$

c) $\sqrt{\left(\frac{4}{9}\right)^2}+\sqrt[3]{-27}-\sqrt{4}=?$

$\sqrt{\left(\frac{4}{9}\right)^2}+\sqrt[3]{-27}-\sqrt{4}=$

$=\sqrt{\left(\frac{2}{3}\right)^2}+\sqrt[3]{(-3)^3}-2=$

$=\frac{2}{3}-3-2=$

$=\frac{2}{3}-5=-4\frac{1}{3}$

d) $\sqrt{9}\cdot\sqrt[3]{(-8)}\cdot\sqrt[5]{-32}=?$

$\sqrt{9}\cdot\sqrt[3]{(-8)}\cdot\sqrt[5]{-32}=$

$=\sqrt{3^2}\cdot\sqrt[3]{(-2)^3}\cdot\sqrt[5]{(-2)^5}=$

$=3\cdot(-2)\cdot(-2)=12$

2) Simplify: $\sqrt{(x-5)^2}+\sqrt{(x+5)^2}$

Solution:

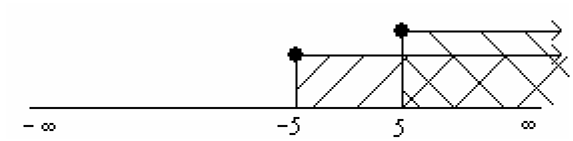
$\sqrt{(x-5)^2}+\sqrt{(x+5)^2}=|x-5|+|x+5|$

How is: $|x-5|=\begin{cases} x-5, & for\ \ x-5\geq 0 \\ -(x-5), & for\ \ x-5<0 \end{cases} = \begin{cases} x-5, & for\ x\geq 5 \\ -(x-5), & for\ x<5 \end{cases}$ and

$|x+5|=\begin{cases} x+5, & for\ \ x+5\geq 0 \\ -(x+5), & for\ \ x+5<0 \end{cases} = \begin{cases} x+5, & for\ x\geq -5 \\ -(x+5), & for\ x<-5 \end{cases}$

First we must see "where are'' solution! There are 4 situacions:

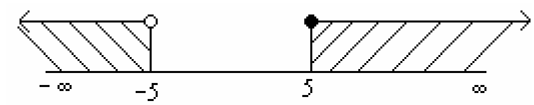
I for $x \ge 5$ i $x \ge -5$



$|x-5|+|x-5| = x-5+x+5 = 2x$

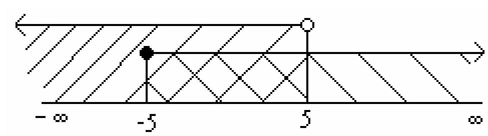
$x \in [5,\infty)$

II for $x \ge 5$ i $x < -5$



No solutions

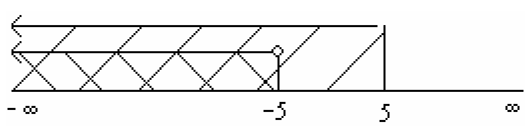
III for $x < 5$ i $x \ge -5$



$|x-5|+|x-5| = -x+5+x+5 = +10$

$x \in [-5,5)$

IV for $x < 5$ i $x < -5$



$|x-5|+|x+5| = -x+5-x-5 = -2x$

$x \in (-\infty,-5)$

Finally: $$|x-5|+|x+5| = \begin{cases} -2x, & x < -5 \\ 10, & -5 \le x < 5 \\ 2x, & x \ge 5 \end{cases}$$

With the characteristics of the root,we can, for the purpose of simplification of expression, remove roots or move them to the desired location.

Examples:

1) $\frac{1}{\sqrt{5}}=\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{5}}{\sqrt{5^2}}=\frac{\sqrt{5}}{5}$

2) $\frac{9}{\sqrt{12}}=\frac{9}{\sqrt{12}}\cdot\frac{\sqrt{12}}{\sqrt{12}}=\frac{9\sqrt{12}}{12}=\frac{3\sqrt{12}}{4}=\frac{3\sqrt{4\cdot 3}}{4}=\frac{3\cdot 2\sqrt{3}}{4}=\frac{3\sqrt{3}}{2}$

3) $\frac{15}{2\sqrt{3}}=\frac{15}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{15\sqrt{3}}{2\cdot 3}=\frac{5\sqrt{3}}{2}$

4) $\frac{6}{\sqrt[3]{2}}=\frac{6}{\sqrt[3]{2}}\cdot\frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}}=\frac{6\cdot\sqrt[3]{2^2}}{\sqrt[3]{2^3}}=\frac{6\cdot\sqrt[3]{4}}{2}=3\sqrt[3]{4}$ because $\sqrt[n]{a^n}=a$, $a>0$

5) $\frac{10}{\sqrt[4]{3}}=\frac{10}{\sqrt[4]{3}}\cdot\frac{\sqrt[4]{3^3}}{\sqrt[4]{3^3}}=\frac{10\sqrt[4]{27}}{\sqrt[4]{3^4}}=\frac{10\sqrt[4]{27}}{3}=$

6) $\frac{ab}{\sqrt[3]{a^2b}}=\frac{ab}{\sqrt[3]{a^2b^1}}\cdot\frac{\sqrt[3]{a^1b^2}}{\sqrt[3]{a^1b^2}}=\frac{ab\sqrt[3]{ab^2}}{\sqrt[3]{a^3b^3}}=\frac{ab\sqrt[3]{ab^2}}{ab}=\sqrt[3]{ab^2}$

By using the formula $(A-B)\cdot(A+B)=A^2-B^2$ we get:

7) $\frac{1}{2+\sqrt{3}}=\frac{1}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{2-\sqrt{3}}{2^2-\sqrt{3}^2}=\frac{2-\sqrt{3}}{4-3}=2-\sqrt{3}$

8) $\frac{11}{\sqrt{6}-\sqrt{2}}=\frac{11}{\sqrt{6}-\sqrt{2}}\cdot\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}=\frac{11(\sqrt{6}+\sqrt{2})}{\sqrt{6}^2-\sqrt{2}^2}=\frac{11(\sqrt{6}+\sqrt{2})}{6-2}=\frac{11(\sqrt{6}+\sqrt{2})}{4}$

9)

$$\frac{5}{2\sqrt{3}-3\sqrt{2}}=\frac{5}{2\sqrt{3}-3\sqrt{2}}\cdot\frac{2\sqrt{3}+3\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}=\frac{5(2\sqrt{3}+3\sqrt{2})}{(2\sqrt{3})^2-(3\sqrt{2})^2}=\frac{5(2\sqrt{3}+3\sqrt{2})}{4\cdot 3-9\cdot 2}=\frac{5(2\sqrt{3}+3\sqrt{2})}{12-18}=\frac{5(2\sqrt{3}+3\sqrt{2})}{-6}$$

How to use formulas?

$$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$
$$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$$

10)

$$\frac{1}{\sqrt[3]{3}+\sqrt[3]{2}} = \frac{1}{\sqrt[3]{3}+\sqrt[3]{2}} \cdot \frac{\sqrt[3]{3^2} - \sqrt[3]{3}\sqrt[3]{2} + \sqrt[3]{2^2}}{\sqrt[3]{3^2} - \sqrt[3]{3}\sqrt[3]{2} + \sqrt[3]{2^2}} = \frac{\sqrt[3]{9} - \sqrt[3]{6} + \sqrt[3]{4}}{\sqrt[3]{3}^3 + \sqrt[3]{2}^3} = \frac{\sqrt[3]{9} - \sqrt[3]{6} + \sqrt[3]{4}}{3+2} = \frac{\sqrt[3]{9} - \sqrt[3]{6} + \sqrt[3]{4}}{5}$$

11)

$$\frac{5}{\sqrt[3]{5}-\sqrt[3]{4}} = \frac{5}{\sqrt[3]{5}-\sqrt[3]{4}} \cdot \frac{\sqrt[3]{5^2} + \sqrt[3]{5}\sqrt[3]{4} + \sqrt[3]{4^2}}{\sqrt[3]{5^2} + \sqrt[3]{5}\sqrt[3]{4} + \sqrt[3]{4^2}} = \frac{5\left(\sqrt[3]{25} + \sqrt[3]{20} + \sqrt[3]{16}\right)}{\sqrt[3]{5}^3 - \sqrt[3]{4}^3} = 5\left(\sqrt[3]{25} + \sqrt[3]{20} + \sqrt[3]{16}\right)$$

12) $\frac{3}{\sqrt[4]{5}-2} = ?$

$$\frac{3}{\sqrt[4]{5}-2} = \frac{3}{\sqrt[4]{5}-2} \cdot \frac{\sqrt[4]{5}+2}{\sqrt[4]{5}+2} = \frac{3\left(\sqrt[4]{5}+2\right)}{\sqrt[4]{5^2}-2^2} = \frac{3\left(\sqrt[4]{5}+2\right)}{\sqrt{5}-4} \cdot \frac{\sqrt{5}+4}{\sqrt{5}+4} = \frac{3\left(\sqrt[4]{5}+2\right)\left(\sqrt{5}+2\right)}{\sqrt[4]{5^2}-4^2} = \frac{3\left(\sqrt[4]{5}+2\right)\left(\sqrt{5}+2\right)}{-11}$$

13) $\frac{6}{\sqrt{21}+\sqrt{7}+2\sqrt{3}+2}$ =?

$$\frac{6}{\sqrt{21}+\sqrt{7}+2\sqrt{3}+2} = \frac{6}{\sqrt{7}\cdot\sqrt{3}+\sqrt{7}+2\left(\sqrt{3}+1\right)} = \frac{6}{\sqrt{7}\left(\sqrt{3}+1\right)+2\left(\sqrt{3}+1\right)}$$

$$= \frac{6}{\left(\sqrt{3}+1\right)\left(\sqrt{7}+2\right)} \cdot \frac{\sqrt{3}-1}{\sqrt{3}-1} \cdot \frac{\sqrt{7}-2}{\sqrt{7}-2} =$$

$$= \frac{6\left(\sqrt{3}-1\right)\left(\sqrt{7}-2\right)}{\left(\sqrt{3}^2-1^2\right)\left(\sqrt{7}^2-2^2\right)} = \frac{6\left(\sqrt{3}-1\right)\left(\sqrt{7}-2\right)}{2\cdot 3} = \left(\sqrt{3}-1\right)\left(\sqrt{7}-2\right)$$

14) $\dfrac{1}{\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}}=?$

$$\frac{1}{\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}}=\frac{1}{\sqrt[3]{3^2}+\sqrt[3]{3}\cdot\sqrt[3]{2}+\sqrt[3]{2^2}}\cdot\frac{\sqrt[3]{3}-\sqrt[3]{2}}{\sqrt[3]{3}-\sqrt[3]{2}}=\frac{\sqrt[3]{3}-\sqrt[3]{2}}{\sqrt[3]{3^3}-\sqrt[3]{2^3}}=\frac{\sqrt[3]{3^3}-\sqrt[3]{2^3}}{3-2}=$$

$$=\frac{\sqrt[3]{3^3}-\sqrt[3]{2^3}}{1}=\sqrt[3]{3^3}-\sqrt[3]{2^3}$$

Lagrange identity:

$$\sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$ where is $a>0,\quad b>0,\quad b<a^2$

Apply it to 2 **examples:** : **a)** $\sqrt{2+\sqrt{3}}$
b) $\sqrt{6-4\sqrt{2}}$

a) $\sqrt{2+\sqrt{3}}=\sqrt{\dfrac{2+\sqrt{2^2-3}}{2}}+\sqrt{\dfrac{2-\sqrt{2^2-3}}{2}}$

$=\sqrt{\dfrac{2+\sqrt{1}}{2}}+\sqrt{\dfrac{2-\sqrt{1}}{2}}$

$=\sqrt{\dfrac{3}{2}}+\sqrt{\dfrac{1}{2}}=\dfrac{\sqrt{3}+1}{\sqrt{2}}$

b) $\sqrt{6-4\sqrt{2}}$ = first we have 4 to insert in the root!=

$$\sqrt{6-\sqrt{16\cdot 2}}=\sqrt{6-\sqrt{32}}=\sqrt{\frac{6+\sqrt{6^2-32}}{2}}+\sqrt{\frac{6-\sqrt{6^2-32}}{2}}$$

$$=\sqrt{\frac{6+\sqrt{36-32}}{2}}+\sqrt{\frac{6-\sqrt{36-32}}{2}}$$

$$=\sqrt{\frac{6+2}{2}}+\sqrt{\frac{6-2}{2}}=\sqrt{4}-\sqrt{2}=2-\sqrt{2}$$

15) Prove that the value of expression $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2\sqrt{3}}}$ **is irrational number.**

Proof:

$2+\sqrt{3}=(\text{previous task})=\dfrac{\sqrt{3}+1}{\sqrt{2}}$, and $2-\sqrt{3}=\dfrac{\sqrt{3}-1}{\sqrt{2}}$, So:

$$\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2\sqrt{3}}}=\frac{2+\sqrt{3}}{\sqrt{2}+\dfrac{\sqrt{3}+1}{\sqrt{2}}}+\frac{2-\sqrt{3}}{\sqrt{2}-\dfrac{\sqrt{3}-1}{\sqrt{2}}}=$$

$$=\frac{2+\sqrt{3}}{\dfrac{2+\sqrt{3}+1}{\sqrt{2}}}+\frac{2-\sqrt{3}}{\dfrac{2-\sqrt{3}+1}{\sqrt{2}}}=$$

$$=\frac{\sqrt{2}\left(2+\sqrt{3}\right)}{3+\sqrt{3}}+\frac{\sqrt{2}\left(2-\sqrt{3}\right)}{3-\sqrt{3}}$$

$$=\frac{\left(2\sqrt{2}+\sqrt{6}\right)\left(3-\sqrt{3}\right)+\left(2\sqrt{2}-\sqrt{6}\right)\left(3+\sqrt{3}\right)}{\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)}$$

$$=\frac{6\sqrt{2}\cancel{-2\sqrt{6}}\cancel{+3\sqrt{6}}-\sqrt{18}+6\sqrt{2}+\cancel{2\sqrt{6}}\cancel{-3\sqrt{6}}-\sqrt{18}}{3^2-\sqrt{3}^2}$$

$$=\frac{12\sqrt{2}-2\sqrt{18}}{9-3}=\frac{12\sqrt{2}-2\sqrt{9\cdot 2}}{6}=\frac{12\sqrt{2}-6\sqrt{2}}{6}=\frac{6\sqrt{2}}{6}=\sqrt{2}$$

16) Prove that: $\dfrac{4+2\sqrt{3}}{\sqrt[3]{10+6\sqrt{3}}}=\sqrt{3}+1$

Proof:

We will go from the left side to get right.

$$4+2\sqrt{3}=3+1+2\sqrt{3}=3+2\sqrt{3}+1=\left(\sqrt{3}\right)^2+2\sqrt{3}+1=\left(\sqrt{3}+1\right)^2$$

$$\left(A+B\right)^3=A+3A^2B+3AB^2+B^3$$

$$\left(\sqrt{3}+1\right)^3=\sqrt{3}^3+3\cdot\sqrt{3}^2\cdot 1+3\cdot\sqrt{3}\cdot 1^2+1^3$$

$$=\sqrt{27}+3\cdot 3\cdot 1+3\sqrt{3}+1$$

$$=\sqrt{9\cdot 3}+9+3\sqrt{3}+1=3\sqrt{3}+3\sqrt{3}+10=10+6\sqrt{3}$$

So: $\dfrac{4+2\sqrt{3}}{\sqrt[3]{10+6\sqrt{3}}}=\dfrac{\left(\sqrt{3}+1\right)^2}{\sqrt[3]{\left(\sqrt{3}+1\right)^3}}=\dfrac{\left(\sqrt{3}+1\right)^2}{\sqrt{3}+1}=\sqrt{3}+1$

17) Simplify: **a)** $\sqrt[3]{x^2\sqrt{x^{-1}}}\cdot\sqrt[3]{x^{-1}\sqrt{x}}$

b) $\sqrt{x\sqrt[3]{x^2}}\cdot\sqrt[3]{x^2}:\left(\sqrt{x^{-1}}\right)^3$

Solution: a)

$$\sqrt[3]{x^2\sqrt{x^{-1}}}\cdot\sqrt[5]{x^{-1}\sqrt{x}}=\sqrt[3]{x^2}\sqrt[3]{\sqrt{x^{-1}}}\cdot\sqrt[5]{x^{-1}}\sqrt[5]{\sqrt{x^{-1}}}=x^{\frac{2}{3}}\cdot\sqrt[6]{x^{-1}}\cdot x^{\frac{1}{5}}\cdot\sqrt[10]{x}=x^{\frac{2}{3}}\cdot x^{-\frac{1}{6}}\cdot x^{-\frac{1}{5}}\cdot x^{\frac{1}{10}}=$$

$$=x^{\frac{2}{3}-\frac{1}{6}-\frac{1}{5}+\frac{1}{10}}=x^{\frac{20-5-6+3}{30}}=x^{\frac{12}{30}}=x^{\frac{2}{5}}=\sqrt[5]{x^2}$$

b)

$$\sqrt{x\sqrt[3]{x^2}}\cdot\sqrt[3]{x^2}:\left(\sqrt{x^{-1}}\right)^3=\sqrt{x}\sqrt{\sqrt[3]{x^2}}\cdot x^{\frac{2}{3}}:\sqrt{x^{-3}}=$$

$$x^{\frac{1}{2}}\cdot x^{\frac{2}{6}}\cdot x^{\frac{2}{3}}:x^{-\frac{3}{2}}=x^{\frac{1}{2}+\frac{2}{6}+\frac{2}{3}-\left(-\frac{3}{2}\right)}=x^{\frac{3+2+4+9}{6}}=x^{\frac{18}{6}}=x^3$$